Comparison of Gross & Harris and Jain CDF Formulas

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In *Optimizing Oracle Performance* [Millsap & Holt (2003): O'Reilly], I noted an incorrectness in the Jain CDF Formula (p236). Frank Ives has kindly provided work showing that the inclusion of a single pair of parenthesis makes the Jain formula equivalent to the Gross & Harris formula. This notebook is a further validation that the Gross & Harris formula and the Jain formula, with Ives's modification, indeed have identical values for all input values, as long as $C = A$ (which is just the "stable queueing system" assumption).

Note also that neither the Jain formula nor the Gross & Harris formula is strictly correct unless you also add the following qualifying condition:

$$\rho \neq \frac{m-1}{m}.$$

Without this qualifying condition, both formulas have a singularity. For example, neither formula can produce a value for $m = 4$, $\rho = .75$. This singularity *is* mentioned in the book. The VisualBasic code for the CDFr function shown on page 235 uses a special conditional block to interpolate the CDF value at the singularity from the two domain values $\rho = \frac{m-1}{m} \pm \epsilon$.

```
Clear["Global`*"];
λ := a / t;
τ := 1 / λ;
x := c / t;
s := b / c;
μ := 1 / s;
u := b / t ;
ρ := u / m;
c := a ; (* ...on a stable queueing system. *)
```

$$\texttt{erlangc} := \frac{\frac{(m\,\rho)^m}{m!}}{(1-\rho)\sum_{k=0}^{m-1}\frac{(m\,\rho)^k}{k!}+\frac{(m\,\rho)^m}{m!}};$$

$$\texttt{p0} := \left(\sum_{n=0}^{m-1}\frac{(m\,\rho)^n}{n!}+\frac{(m\,\rho)^m}{m!\,(1-\rho)}\right)^{-1};$$

$$\texttt{Wq0} := 1-\frac{(m\,\rho)^m\,\texttt{p0}}{m!\,(1-\rho)};$$

$$\text{jain} = 1 - e^{-\mu r} - \frac{\text{erlangc}}{1 - m + m\,\rho}\left(e^{-m\,\mu\,(1-\rho)\,r} - e^{-\mu\,r}\right);$$

$$\text{gross} = \frac{m\,(1-\rho) - \text{Wq0}}{m\,(1-\rho) - 1}\left(1 - e^{-\mu\,r}\right) - \frac{1 - \text{Wq0}}{m\,(1-\rho) - 1}\left(1 - e^{-(m\,\mu-\lambda)\,r}\right);$$

jain – gross

$$1 - e^{-\frac{a\,r}{b}} + \frac{\left(1 - e^{r\left(-\frac{a\,m}{b} + \frac{a}{t}\right)}\right)\left(\frac{b}{t}\right)^m}{\left(-1 + m\left(1 - \frac{b}{m\,t}\right)\right)\left(1 - \frac{b}{m\,t}\right) m!\left(\frac{\left(\frac{b}{t}\right)^m}{\left(1-\frac{b}{m\,t}\right) m!} + \frac{e^{b/t}\,\text{Gamma}\left[m, \frac{b}{t}\right]}{\text{Gamma}[m]}\right)} -$$

$$\frac{\left(-e^{-\frac{a\,r}{b}} + e^{-\frac{a\,m\,r\left(1-\frac{b}{m\,t}\right)}{b}}\right)\left(\frac{b}{t}\right)^m}{\left(1 - m + \frac{b}{t}\right) m!\left(\frac{\left(\frac{b}{t}\right)^m}{m!} + \frac{e^{b/t}\left(1-\frac{b}{m\,t}\right)\text{Gamma}\left[m, \frac{b}{t}\right]}{\text{Gamma}[m]}\right)} -$$

$$\frac{\left(1 - e^{-\frac{a\,r}{b}}\right)\left(-1 + m\left(1 - \frac{b}{m\,t}\right) + \frac{\left(\frac{b}{t}\right)^m}{\left(1-\frac{b}{m\,t}\right) m!\left(\frac{\left(\frac{b}{t}\right)^m}{\left(1-\frac{b}{m\,t}\right) m!} + \frac{e^{b/t}\,\text{Gamma}\left[m, \frac{b}{t}\right]}{\text{Gamma}[m]}\right)}\right)}{-1 + m\left(1 - \frac{b}{m\,t}\right)}$$

Simplify[%]

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